Last time:

* Maxwell's equations:

(i) $\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho$

(ii) $\nabla \cdot \underline{B} = 0$

(iii) $\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$

(iv) $\nabla \times \underline{E} - \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}$

* Uncoupling these equations: (in vacuum)

$$\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2} \quad ; \quad \nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$

$\underline{E}$ & $\underline{B}$ satisfy the wave equation

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$ where v is the velocity of wave propagation

$$v = \frac{1}{(\varepsilon_0 \mu_0)^{1/2}} = 3 \times 10^8 \, m/s \equiv c \qquad (E = mc^2)$$

Mathematically what is the wave equation?

$$\nabla^2 f = \frac{1}{v} \frac{\partial^2 f}{\partial t^2}$$ we can verify that $f = f(x \pm vt)$ is a solution to this eq.

For 1D:

Let $u = x \pm vt \Rightarrow \frac{\partial u}{\partial x} = 1$ and $\frac{\partial u}{\partial t} = \pm v$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v} \frac{\partial^2 f}{\partial t^2}$$

we'll calculate the Lhs:

$$\frac{\partial f(u)}{\partial x} = \frac{\partial f}{\partial u} \cancel{\frac{\partial u}{\partial x}}^{1} = \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(u)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial^2 f}{\partial u^2} \cancel{\frac{\partial u}{\partial x}}^{1}$$
$$= \frac{\partial^2 f}{\partial u^2} \quad (*)$$

Now we calculate the Rhs:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cancel{\frac{\partial u}{\partial t}}^{\pm v} = \pm v \frac{\partial f}{\partial u}$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\pm v \frac{\partial f}{\partial u}\right) = \pm v \frac{\partial^2 f}{\partial u^2} \cancel{\frac{\partial u}{\partial t}}^{\pm v} = v^2 \frac{\partial^2 f}{\partial u^2} \quad (**)$$

Subs this in the wave eq:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$ subs (*) & (**)

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{v^2}\left(\cancel{v^2}\,\frac{\partial^2 f}{\partial w^2}\right) \Rightarrow \text{identity}$$

$\therefore f = f(x \pm vt)$ is a solution to the wave equation

Since this is a linear differential equation

If f_1 & f_2 are solutions $\Rightarrow f = f_1 \pm f_2$ is also a solution

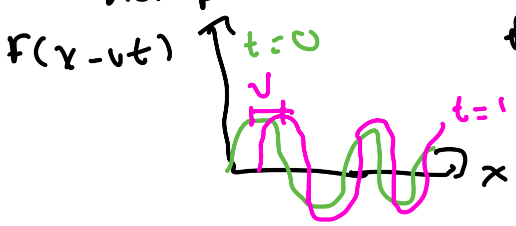
In 1D the most general wave eq is:

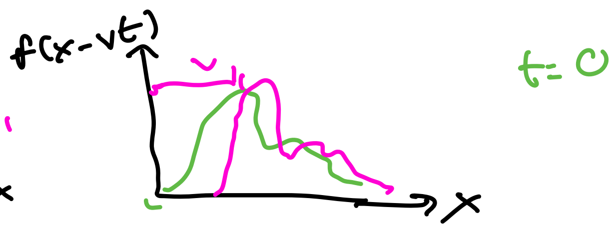
$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}$$ with solution $$\psi(x,t) = f(x-vt) + g(x+vt)$$

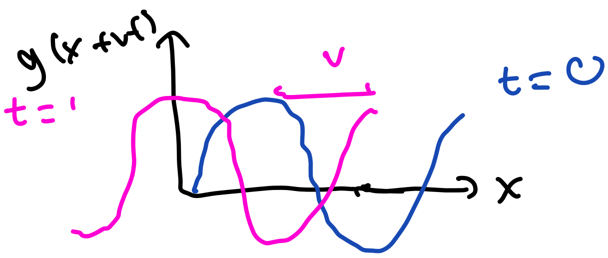
If x is a position and t is time:

- $f(x-vt)$
  - wave moving with velocity v in the $+x$ direction.
  - Profile of f will be shifted by an amount v at every time step
- $g(x+vt)$
  - wave moving with velocity v in the $-x$ direction
  - Profile of f will be shifted by an amount v to the left at every time step.

f and g describe traveling waves, they can be entirely irregular not periodic







Standing wave

Nodes stay the same $\rightarrow$ superposition of 2 waves traveling in opposite directions

$$f(x,t) = A_1 \sin(k_1 x - \omega_1 t)$$
$$g(x,t) = A_2 \sin(k_2 x + \omega_2 t)$$

$$k_i = \frac{2\pi}{\lambda_i} = \frac{2\pi f_i}{v_i} = \frac{\omega_i}{v_i} \equiv \text{wave number}$$

For simplicity
$A_1 = A_2 = 1$
$k_1 = k_2 = k$

$w_1 = w_2 = w$

$f(x,t) = \sin(kx - wt)$
$g(x,t) = \sin(kx + wt)$

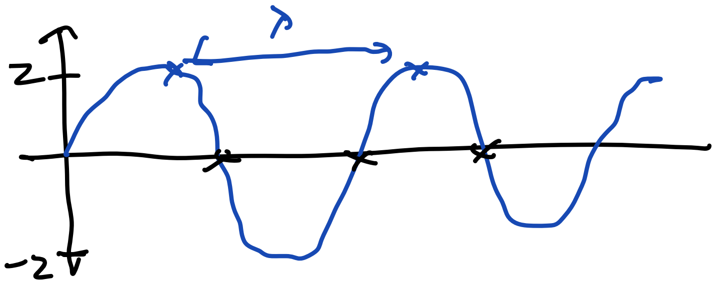
The solution to the wave eq becomes

$$\psi(x,t) = \sin(kx - wt) + \sin(kx + wt)$$

Using identity
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$\psi(x,t) = 2\sin kx \cos wt$ ← a standing wave.

The time evolution of the wave's profile in x is an oscillation in amplitude but the wave pattern does not move



Plane waves

For $\underline{E}$ & $\underline{B}$ they are solutions to the wave eq in 3D

$$\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2} \quad ; \quad \nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$

in cartesian
$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$$

For 1D
$f(x \pm vt)$

For 3D
$f(\underline{r} \pm v\hat{k}t)$
↑ direction of wave propagation

We'll rewrite the argument of f: $\underline{r} \pm v\hat{k}t$ adding the wave vector $\underline{k}$

$$\underline{k} \cdot (\underline{r} \pm v\hat{k}t) = \underline{k} \cdot (\underline{r} \pm \frac{w}{k}\hat{k}t) = \underline{k} \cdot \underline{r} \pm wt$$

here
$w = vk$
$w = 2\pi\nu$

we have such solutions

$f(\underline{k} \cdot \underline{r} \pm wt)$ these are periodic functions

Sines and cosines are the building blocks of all waves.

Plane waves can be written as:

$$\underline{E} = \underline{E}_0 \sin(\underline{k} \cdot \underline{r} - wt) = \underline{E}_0 \sin(k_x x + k_y y + k_z z - wt)$$

$$\underline{B} = \underline{B}_0 \sin(\underline{k} \cdot \underline{r} - wt) = \underline{B}_0 \sin(k_x x + k_y y + k_z z - wt)$$

$\underline{k}$ = wavevector

$|\underline{k}|$ = wavenumber

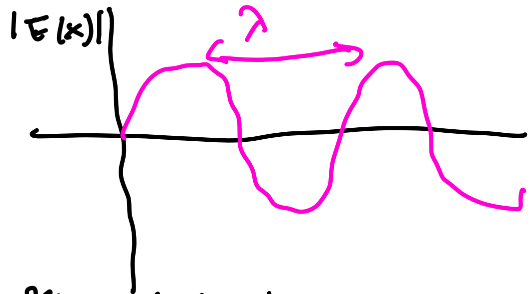
$\hat{k}$ = direction of propagation.

We will choose a coordinate system such that $\underline{k}$ is oriented // to the $\hat{x}$ axis, in the case of the solution for $\underline{E}$:

$$\underline{E} = \underline{E}_0 \sin(k_x x - \omega t)$$

① Consider $t=0$. Looking at the spatial variation

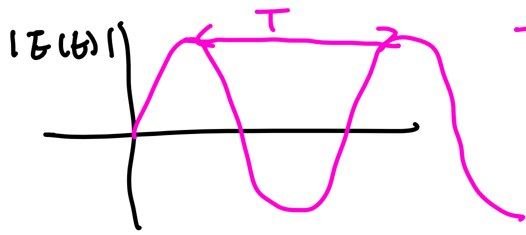
$$\underline{E} = \underline{E}_0 \sin(k_x x)$$



λ = distance between peaks

② Consider $x=0$. Looking at the temporal variation

$$\underline{E} = \underline{E}_0 \sin(\omega t)$$



T = time between 2 peaks.

velocity $= \frac{\lambda}{T} = \lambda\nu = c$ ← speed of light

$\omega = \frac{2\pi}{T} = 2\pi\nu$; $\omega = ck$; $\lambda\nu = c$

ν ↑ frequency

What types of plane waves satisfy Maxwell's equations?

(i) $\nabla \cdot \underline{E} = 0$ We calculate this for the general solution of the ~ eq

$$\nabla \cdot \underline{E} = \nabla \cdot [\underline{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)]$$

$$= (\underline{E}_{0x} k_x + \underline{E}_{0y} k_y + \underline{E}_{0z} k_z) \cos(k_x x + k_y y + k_z z - \omega t)$$

$$= \underline{k} \cdot \underline{E}_0 \cos(\underline{k}\cdot\underline{r} - \omega t) = 0$$ What does this mean for KAI?

$\nabla \cdot \underline{E} = 0$ when $\underline{k} \cdot \underline{E}_0 = 0$ ⇒ $\boxed{\underline{E} \text{ is } \perp \text{ to } \underline{k}}$ the propagation direction

(ii) $\nabla \cdot \underline{B} = 0$

$$\nabla \cdot \underline{B} = \underline{k} \cdot \underline{B}_0 \cos(\underline{k}\cdot\underline{r} - \omega t) = 0$$

when $\underline{k} \cdot \underline{B}_0 = 0$ ⇒ $\boxed{\underline{B} \perp \text{ to direction of propagation}}$

(iii) $\nabla \times \underline{E} = \mu_0 \varepsilon_0 \frac{\partial \underline{B}}{\partial t}$

$\frac{\partial}{\partial t}$ doesn't change the direction of $\underline{B}$

$\underline{B}$ points in the direction of $\nabla \times \underline{E}$, which is $\perp$ to the plane of circulation of $\underline{E}$

$\Rightarrow \boxed{\underline{E} \perp \underline{B}}$

$\therefore \underline{k} \perp \underline{B} \perp \underline{E}$ EM plane waves are transverse since $\underline{E}$ & $\underline{B}$ are $\perp$ to the direction of propagation

(iv) $\nabla \times \underline{B} = \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$ $\Rightarrow \underline{E} \perp \underline{B}$

We'll calculate the LHS & RHS of this eq:

First the RHS

$$\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \mu_0 \varepsilon_0 \cos(\underline{k}\cdot\underline{r} - \omega t)(-\omega) = -\mu_0\varepsilon_0 \omega\, \underline{E}_0 \cos(\underline{k}\cdot\underline{r} - \omega t)$$

For the LHS:

$$\nabla \times \underline{B} = \nabla \times [\underline{B}_0 \sin(k_x x + k_y y + k_z z - \omega t)]$$

$$= (\underline{k} \times \underline{B}_0)\cos(\underline{k}\cdot\underline{r} - \omega t)$$

Now equating RHS = LHS

$$(\underline{k} \times \underline{B}_0)\cos(\underline{k}\cdot\underline{r} - \omega t) = -(\mu_0 \varepsilon_0)^{1/2} k \cos(\underline{k}\cdot\underline{r} - \omega t)\,\underline{E}_0$$

$$\Rightarrow (\underline{k} \times \underline{B}_0) = -\underline{E}_0 (\mu_0 \varepsilon_0)^{1/2} k$$

$$\Rightarrow (\hat{\underline{k}} \times \underline{B}_0) = -\underline{E}_0 (\mu_0 \varepsilon_0)^{1/2}$$ $\Leftarrow$ $\underline{k}$, $\underline{E}$ and $\underline{B}$ are right handed orthogonal vectors.

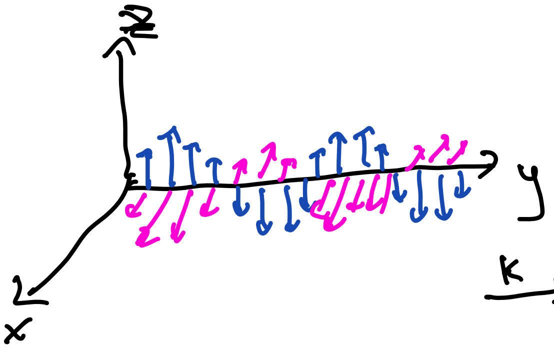
In terms of the magnitude:

$$|\hat{k} \times \underline{B}_0| = |-E_0 (\mu_0 \varepsilon_0)^{1/2}|$$

$$\Rightarrow E_0 = \frac{\pm B_0}{(\mu_0 \varepsilon_0)^{1/2}} \Rightarrow \boxed{E_0 = \pm c B_0}$$

Conclusions:

* Planar EM waves are transverse $\rightarrow$ Direction of $\underline{E}$ and $\underline{B}$ is $\perp$ to the direction of propagation $\underline{k}$
* $\underline{E} \perp \underline{B} \perp \underline{k}$
* Relationship between the magnitudes of E & B is:
  $E_0 = \pm c B_0$
* Planar EM waves are monochromatic. ∓ wave frequencies correspond to one color.



$\underline{E}$ in the z, y plane

$\underline{B}$ in the x, y plane

Purcell p. 439

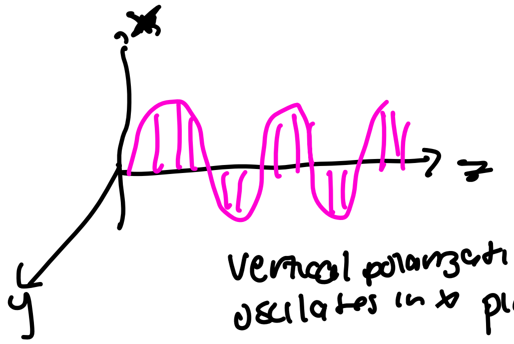
Polarization of EM waves

EM waves are transverse ↔ their direction of propagation is ⊥ to $\underline{E}$ & $\underline{B}$

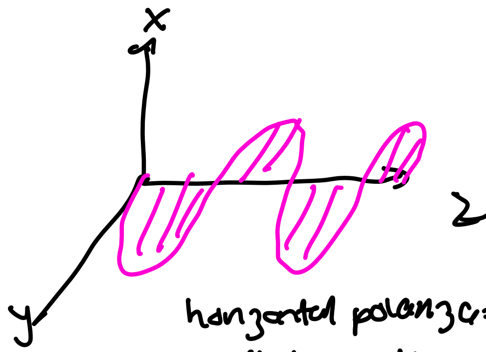
⇔ their direction of propagation is ⊥ to the plane in which $\underline{E}$ & $\underline{B}$ oscillate.

Polarization is the property of transverse waves that specifies the geometrical orientation of oscillations

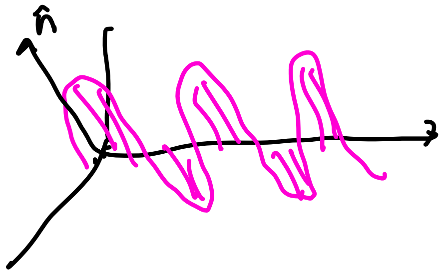
Look at Griffiths p 392



vertical polarization oscillates in x plane



horizontal polarization oscillates in the y plane



$\hat{n}$ gives direction of polarization

Polarization of EM waves

If the directions of $\underline{E}_0$ and $\underline{B}_0$ are constant in time → wave is linearly polarized

If the directions of $\underline{B}_0$ and $\underline{E}_0$ rotate at a frequency ω → wave is circularly polarized.

linearly polarized wave

$$\underline{E} = E_0 \cos(kx - \omega t)\hat{y}$$
$$\underline{B} = B_0 \cos(kx - \omega t)\hat{z}$$

circularly polarized wave

$$\underline{E} = E_0 \hat{x} \sin(kz - \omega t) + E_0 \hat{y} \cos(kz - \omega t)$$
$$\underline{B} = B_0 \hat{y} \sin(kz - \omega t) - B_0 \hat{x} \cos(kz - \omega t)$$

By convention the direction of polarization of an EM wave is the polarization of $\underline{E}$.

Polarization of random light

Light from a bulb, the sun, etc is NOT polarized. It's a superposition of many plane waves.

$$E_{random} = \sum_i E_0 (\hat{x}\cos\theta_i + \hat{y}\sin\theta_i)\cos(kz - \omega t)$$

We can polarize light using a polarizer.

Polaroids: sheets of materials with organic molecules aligned in one direction ⇒ They can carry current in one direction.

When linearly polarized light hits the material:

- If $\underline{E}$ is aligned w/ the direction of the molecules charges will move through it an a current will be generated. $\bar{e}$ moving in this current will collide so "light is stopped".

* If $\underline{E}$ is ⊥ to the orientation of the molecules charges won't move → an light will go through.

Preferred direction ≡ axis along which polarizer transmits light.

$E_\perp$ preferred direction → transparent

$E_\parallel$ preferred direction → no light goes through.

What happens when light is polarized in a direction in between the preferred direction and ⊥?

Light will go through partially.

If light is polarized along $\vec{x}$ and the polaroid is oriented at an angle θ:

$$\underline{E} = E_0 \cos(kz - \omega t)\hat{x}$$

$$\vec{P} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

Field will be an overlap between incoming $\underline{E}$ & polaroid's orientation

$$|E_{out}| = \underline{E}\cdot\vec{P} = E_0\cos(kz-\omega t)(\hat{x}\cos\theta + \hat{y}\sin\theta)\cdot\vec{x}$$

$$= E_0 \cos\theta \cos(kz - \omega t)$$

Polaroid reduces the amplitude of incident light by $\cos\theta$